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RELAXATION METHODS APPLIED TO ENGINEERING PROBLEMS

VIIC. FREE TRANSVERSE VIBRATIONS OF MEMBRANES, WITH AN APPLICATION (BY ANALOGY) TO TWO-DIMENSIONAL OSCILLATIONS IN AN ELECTRO-MAGNETIC SYSTEM

By D. N. de G. Allen, L. Fox, H. Motz and R. V. Southwell, F.R.S.

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Membranes whether uniform or non-uniform in density are easily treated by a technique similar to that of Part VIIB, and with an accuracy more than sufficient for practical purposes. An equation of the same mathematical form governs certain practically-important types of high-frequency electromagnetic oscillation, and here the illustrative example treated has direct importance for design.

Introductory

1. Part VI of this series extended the relaxation technique to determine characteristic modes and frequencies of freely vibrating systems, Part VII B applied similar devices to the problem of elastic stability for a flat plate sustaining forces in its plane*. With slight modification the same methods can be employed to determine modes and frequencies of flat plates executing free transverse vibrations, so this problem too may be said to have been covered in the series. But no consideration has been given, as yet, to the simpler problem of vibrating flexible membranes. (There can be no question of elastic stability, since a membrane can sustain no thrust.)

Apart from some acoustical applications, little interest attaches to membrane vibrations *per se*. But their governing equation in the case of uniform density, of the form

$$\nabla^2 w + \lambda w = 0 \quad \left(\nabla^2 \equiv \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right), \tag{1}$$

finds important applications in other fields: it is presented in the theory of convective motion due to non-uniform heating (Pellew & Southwell 1940*b*, § 13), and it governs certain types of high-frequency oscillation in electromagnetic systems. Recent developments in the technique of production and detection of ultra-short electromagnetic waves have led to the use of hollow metal tubes and cavities as wave guides and resonators; consequently importance attaches to the vibrations which can occur in such cavities, and methods are required for calculating their natural modes and frequencies. By orthodox methods it has so far only been possible to deal with very simply shaped

* Parts VI and VII B are included in the references as Pellew & Southwell 1940*a*, Christopherson, Fox, Green, Shaw & Southwell 1945.

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cavities (e.g. parallelepipeds, cylinders and spheres), and for many practically important shapes only rough estimates are available. Unrestricted methods having an accuracy (say) of 1 % will thus have considerable value.

2. Relaxation methods were in fact based originally upon the 'membrane analogy' of the plane-potential problem, and their terminology reflects that circumstance. It has therefore seemed appropriate in this paper to treat membrane vibrations as the primary problem, and to bring electromagnetic oscillations under discussion by analogy, notwithstanding that this inverts their order in respect of practical importance. Accordingly Section I derives the governing equation for a membrane having any specified distribution of density, and applies the relaxation technique to find, for a fixed square boundary, the gravest normal mode and frequency. The exact solution is known as regards a membrane of uniform density (Rayleigh 1926, § 197): Section I gives approximate solutions obtained by L. F., for comparison, with a use of the technique developed in Part VII B. It also exemplifies the similar treatment of membranes non-uniform in density, showing that the general case is little harder: here no exact solution is available for comparison, but there is no reason to believe that the accuracy is less. The customary boundary condition (of zero displacement) is imposed in every case.

3. The less usual condition of *zero normal gradient of displacement* is presented in the convection problem mentioned earlier (§ 1), and both types of boundary condition can arise in electromagnetic problems. This application was suggested by H. M., with whom D. N. de G. A. has been associated in the attack by relaxation methods. Section II explains the derivation from Maxwell's equations of the governing equation (1), and briefly summarizes the numerical computations. The way is now open for a more exhaustive treatment of electromagnetic problems by the technique which this paper has explained and tested.

It hardly needs to be stated that attention, in this paper, is confined to free vibrations for the reason that forced oscillations, due to pressures of known distribution and frequency, present a relatively easy problem, tractable by the same technique as applies to cases of static loading. We have, moreover, confined attention here to *gravest* modes: the determination of higher modes and frequencies, with a use of 'conjugate relations' to eliminate unwanted modes from the assumed solution, has been fully explained in earlier publications (Part VI, §§ 14–19; Southwell 1940, Chap. VIII).

I. 'Normal' free vibrations of a uniformly tensioned membrane

The governing equations

4. This section relates to membranes stretched so as to have uniform tension (the same in all directions), but not necessarily uniform in density. The governing equation is easily obtained in the manner of Rayleigh 1926, Chap. IX.*

* Rayleigh's derivation and notation have been modified slightly, to conform with earlier papers in this series and with Southwell 1940, Chap. VII.

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The potential energy of the tension is increased as a consequence of transverse displacement by an amount

$$\left.\begin{aligned}\mathfrak{V} &= \tfrac{1}{2}T_1\iint\left\{\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right\}dx\,dy,\\ &\text{and the kinetic energy is given by}\\ \mathfrak{T} &= \frac{1}{2}\iint \sigma\dot{w}^2\,dx\,dy,\end{aligned}\right\} \tag{i}$$

both surface integrals extending to the whole area of the membrane. (T_1 denotes the line density of the membrane tension, σ its surface density.) In a 'normal' free vibration

$$w = \mathbf{w}\sin(pt+\epsilon), \tag{ii}$$

where $\mathbf{w}$ is a function of x and y (but not of t), p ($=2\pi n$) defines the frequency of vibration, and ϵ is an arbitrary phase-constant. Then

$$\mathfrak{V} = \mathbf{V}\sin^2(pt+\epsilon), \quad \mathfrak{T} = p^2\mathbf{T}\cos^2(pt+\epsilon), \tag{iii}$$

and conservation of energy requires that

$$\mathbf{V} = p^2\,.\,\mathbf{T}, \tag{2}$$

where $$\mathbf{V} = \tfrac{1}{2}T_1\iint\left\{\left(\frac{\partial \mathbf{w}}{\partial x}\right)^2+\left(\frac{\partial \mathbf{w}}{\partial y}\right)^2\right\}dx\,dy, \quad \mathbf{T} = \frac{1}{2}\iint\sigma\,.\,\mathbf{w}^2\,dx\,dy. \tag{3}$$

5. When the mode is known (i.e. $\mathbf{w}$ as a function of x and y), equation (2) serves to determine the frequency constant p^2. But it is known, further, that *for a normal vibration p^2 as calculated from* (2) *is stationary for all variations of the mode*, and the governing equation can be deduced from this circumstance; for we have

$$\delta\mathbf{V}-p^2\,.\,\delta\mathbf{T} = \mathbf{T}\,.\,\delta p^2 = 0,$$

therefore

$$\begin{aligned}p^2\iint\sigma\,.\,\mathbf{w}\,.\,\delta\mathbf{w}\,dx\,dy &= T_1\iint\left(\frac{\partial\mathbf{w}}{\partial x}\cdot\frac{\partial}{\partial x}\delta\mathbf{w}+\frac{\partial\mathbf{w}}{\partial y}\cdot\frac{\partial}{\partial y}\delta\mathbf{w}\right)dx\,dy\\ &= T_1\oint\delta\mathbf{w}\,.\,\frac{\partial\mathbf{w}}{\partial n}\,ds - T_1\iint\delta\mathbf{w}\,.\,\nabla^2\mathbf{w}\,dx\,dy,\end{aligned} \tag{iv}$$

∇^2 having the significance given in (1), and the line integral extending to the whole of the boundary. Now in all cases that we consider in this paper either $\mathbf{w}$ (and therefore its variation $\delta\mathbf{w}$) or $\dfrac{\partial\mathbf{w}}{\partial n}$ has to vanish at every point of the boundary: consequently the line integral is zero in (iv), which accordingly requires that

$$\iint\delta\mathbf{w}(\sigma p^2\mathbf{w}+T_1\nabla^2\mathbf{w})\,dx\,dy = 0 \tag{v}$$

for all variations $\delta\mathbf{w}$, and so yields

$$T_1\nabla^2\mathbf{w}+\sigma p^2\mathbf{w} = 0 \tag{4}$$

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as the governing equation which must be satisfied at every point in the membrane. In this first section we shall take the boundary condition to be

$$\mathbf{w} = 0. \tag{5}$$

When the surface density σ is uniform, (4) has the form of equation (1). In acoustical problems the density is usually but not always uniform, and since non-uniformity adds little to the labour of a relaxation treatment it has seemed worth while to maintain generality here.

'Non-dimensional' equations

6. As in all applications of the relaxation method, the first step must be to eliminate 'dimensional' factors. Using L to denote some representative dimension of the membrane, we write

$$x' \text{ for } x/L, \quad y' \text{ for } y/L, \quad \rho \text{ for } \sigma/\sigma_0, \quad \lambda \text{ for } \sigma_0 p^2 L^2/T_1, \tag{6}$$

where σ_0 stands for the density at some specified point. Then (4) reduces to

$$\left.\begin{aligned} \nabla'^2\mathbf{w}+\lambda\rho\mathbf{w} &= 0,\\ \text{simply, where} \qquad \nabla'^2 &\equiv \frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}. \end{aligned}\right\} \tag{7}$$

The magnitude of $\mathbf{w}$ is immaterial, as in all cases of normal free vibration; all other quantities which appear in (7) are purely numerical. When σ is uniform, $\rho = 1$ and (7) becomes identical in form with (1).

We also require 'non-dimensional' forms of (2) and (3). On substitution from (6) these yield the relation

$$\left.\begin{aligned} &\mathbf{V}' = \lambda\,.\,\mathbf{T}',\\ \text{where} \qquad &\mathbf{V}' = \iint\left\{\left(\frac{\partial\mathbf{w}}{\partial x'}\right)^2+\left(\frac{\partial\mathbf{w}}{\partial y'}\right)^2\right\}dx'dy', \qquad \mathbf{T}' = \iint\rho\mathbf{w}^2\,dx'dy'. \end{aligned}\right\} \tag{8}$$

Henceforward we shall suppress the dashes in (7) *and* (8), *so* x, y, $\mathbf{w}$, $\mathbf{V}$ *and* $\mathbf{T}$ *are to be regarded as purely numerical quantities defined by* (7) *and* (8) *thus modified.*

Principles of the relaxation treatment

7. These will be taken, *mutatis mutandis*, from §§ 20–5 of Part VII B—with this simplification, that no occasion will arise in the examples treated here for the device of 'optimal synthesis' (§ 23). A membrane, subject to the boundary condition (5), is unlike the flat plate with its double boundary condition, in that the gravest mode is invariably characterized by an absence of nodal lines (other than the boundary). Here we are interested in the gravest mode, which accordingly can be guessed with fair accuracy; so we need not anticipate 'regression' (Part VII B, § 22) to a mode which is not wanted. The point will be made plain in the examples which follow.

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Starting with an assumed form *in which* **w** *is one-signed* throughout, we compute **V** and **T**, and hence deduce a starting estimate of λ, according to (8); then, for this value of λ, we compute residual forces from the finite-difference approximation to

$$\mathbf{F} = \nabla^2\mathbf{w} + \lambda\rho\mathbf{w}, \tag{9}$$

which corresponds with (7) *when the dashes are suppressed* (§ 6). As in previous papers we employ the approximation

$$\frac{a^2}{4}(\nabla^2 w)_0 \approx \frac{1}{N}\Sigma_{a,N}(w) - w_0,$$

in which a (purely numerical) stands for the 'mesh-side', N for the number of points which are joined symmetrically with the typical point 0, and $\Sigma_{a,N}(w)$ for the sum of the w-values at these points. ($N = 4$ for a square, 6 for a triangular net.)

The residual forces are now 'liquidated' (partially) with the aid of 'relaxation patterns' deduced from (9) in accordance with the procedure now become standard (Part VII A, § 17). Then, a new estimate of λ is deduced from (8) with our estimate of **w** thus improved; further liquidation follows (with patterns corresponding with this closer estimate of λ); and so on until λ no longer alters appreciably as the result of this cycle of operations.

As in Parts III and VII, employment of finite-difference approximations means that a net of finite mesh is substituted for the continuous membrane, and the extent of the error thereby introduced is left for judgement by intuition (manifestly it decreases with the size of mesh): correspondingly, the double integrals in (8) are replaced by double summations based on approximations of the type of 'Simpson's rule'. Thereby we obtain as a convenient parallel to (9)

$$\mathbf{F} = \Sigma_{a,N}(\mathbf{w}) + N\left(\frac{\lambda\rho a^2}{4} - 1\right)\mathbf{w}. \tag{10}$$

(We have multiplied (9) by $Na^2/4$, as is legitimate since our aim will be to reduce **F** everywhere to zero.)

Example 1. *Membrane having a square boundary which is fixed*

8. It is natural to take L, in (6), as the side of the square boundary. We know (cf., for example, Rayleigh 1926, § 197) that the gravest value of λ is $2\pi^2 = 19{\cdot}73921$: consequently we can assess the accuracy of our method by this example, which (on account of the 8-fold symmetry of the fundamental mode) entails little difficulty, treated on the basis of *square nets*.

The computations (by L. F.) were started on a square net of mesh-side $a = 1/16$, with an initial assumption which was intentionally made wide of the truth—namely, displacement varying with position like the height of a symmetrical pyramid having the square as base. For this assumption equation (8) gave 23·808 as the starting value

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of λ; but three cycles of liquidation, performed in the manner of § 7, brought the value quickly down to 19·652—the accepted value for this first net. That an underestimate results from an application of Rayleigh's principle is due, of course, to the incidental employment of approximate formulae for double integration. The figure given above was obtained with a (repeated) use of 'Simpson's rule'.

9. As an alternative, the '8-strip formula' given by Bickley (1939) was used with a result ($\lambda = 19{\cdot}841$) which is *less near to the truth*. This being thought somewhat surprising, Bickley's 'four-strip formula' was applied to a test case given by Kármán & Biot (1940)—namely, the integral

$$4\int_0^1 \frac{dx}{1+x^2} = \pi = 3{\cdot}14159265.$$

It gave the result 3·14212, whereas Simpson's rule used with only four divisions in the range gives 3·14157. (Kármán & Biot give results to nine significant figures.) Thus here too a better result is obtained by a use of the simpler formula.

It would seem that the relative accuracy of Simpson's rule and of more elaborate formulae is a matter calling for further study. The error of each formula has of course been stated, but only on the understanding that the wanted function can be identified with a polynomial: the practical question is whether better results will not be obtained by a use of Simpson's rule, notwithstanding that in each pair of subdivisions ('two strips') it identifies the wanted function with a quadratic function, because in its application to computed values *it is not bound by conditions of continuity*.

10. The computations were continued on a finer net ($a = 1/32$) with results which are recorded in figure 1. Very little additional labour was entailed, and the finally accepted value of λ (namely, 19·7200) indicates that the procedure suggested in § 7 will yield results having amply sufficient accuracy for practical purposes. (The mode as given in figure 1 is also very accurate.) Simpson's rule was employed to evaluate the double integrals.

Example 2. *The same problem for a membrane non-uniform in density*

11. Figure 1 was used as the starting assumption in an attack (also by L. F.) on the same problem modified by non-uniformity of the surface density σ. The variation was taken to be given by

$$\rho = \sigma/\sigma_0 = 1+4xy, \tag{11}$$

σ_0 relating to the middle point of the square, and x and y having 'non-dimensional' significance (§ 6). This expression was chosen as being simple to evaluate at the nodal points. It vanishes at two of the four corners, but is positive at every point which is free to move.

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The corresponding value of λ was 19·7200 as before—an obvious consequence of the symmetries possessed by σ and by the assumed mode; but for this value the residual forces had the sign of the change in density, i.e. of xy. With very little labour, point relaxation led to the altered mode of figure 2 (in which fine lines give, for comparison,

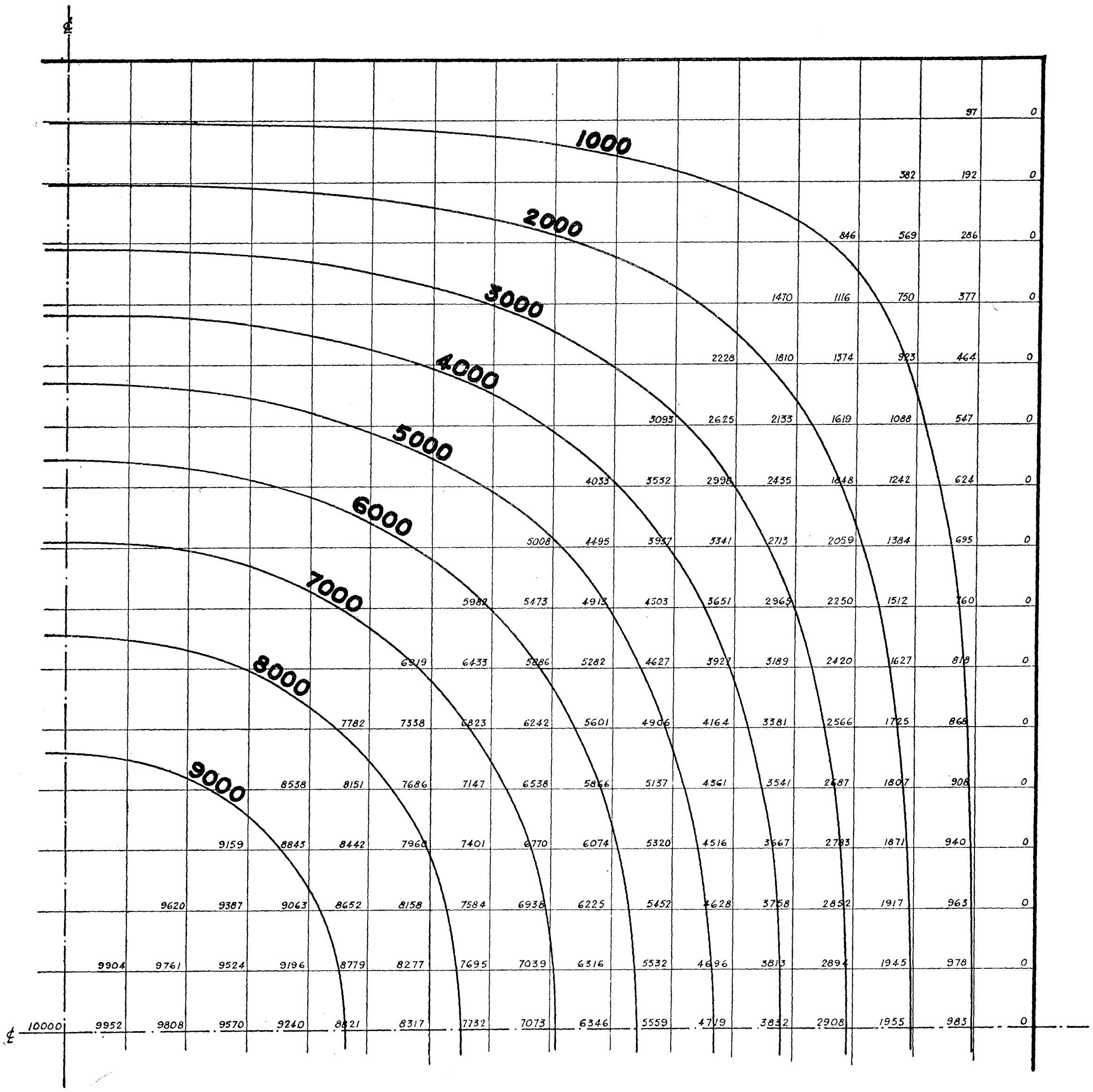

FIGURE 1

contours from figure 1). As was to be expected, the denser parts of the membrane deflect more, the lighter parts less than before. The finally accepted value of λ was 19·5661, and it seems safe to conclude (on the basis of our result for Example 1) that the true value lies between 19·54 and 19·60 (it was to be expected that the value would be lower in this example). For practical purposes this is more than sufficient accuracy.

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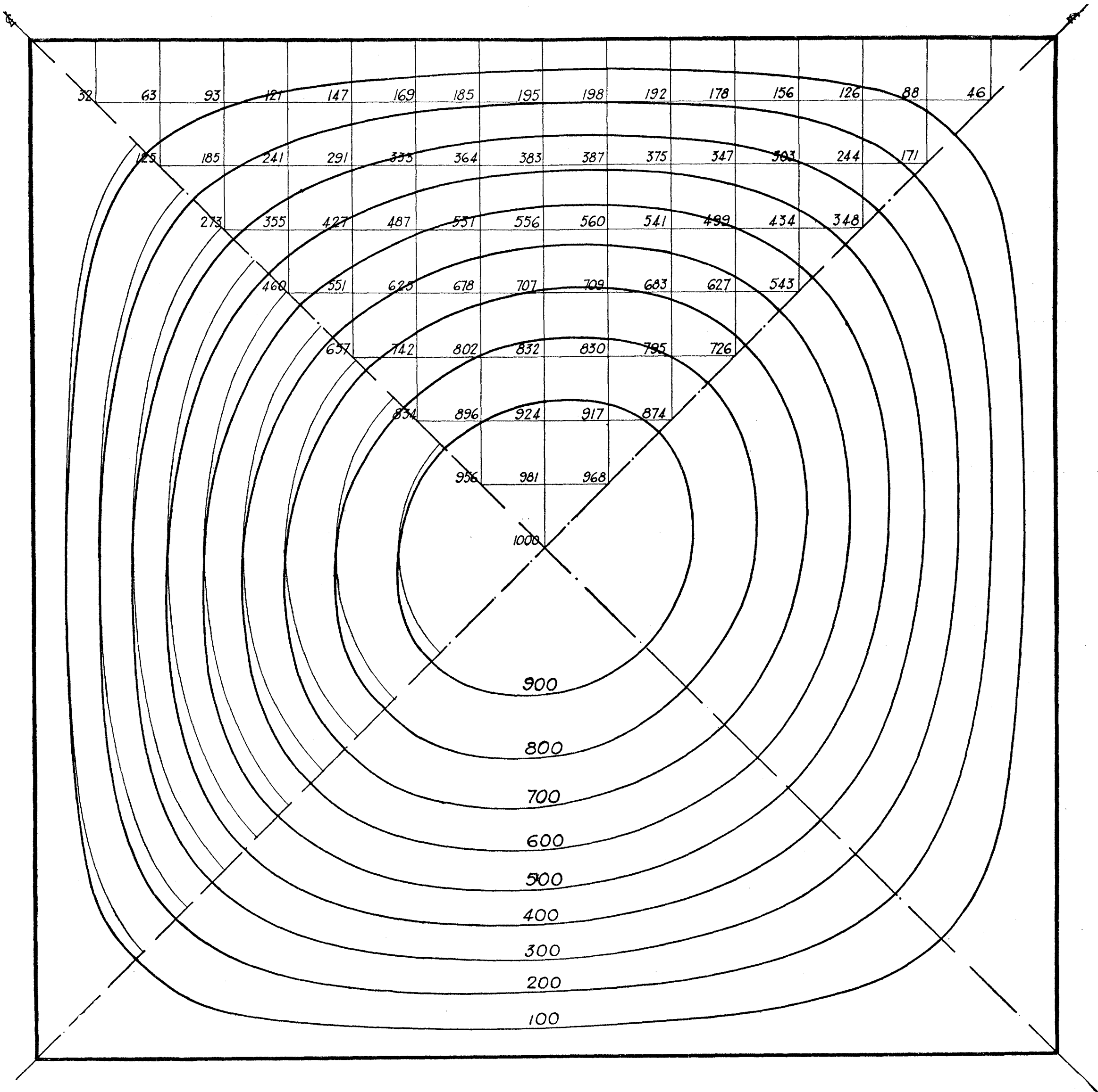

FIGURE 2

II. TWO-DIMENSIONAL ELECTROMAGNETIC OSCILLATIONS WHICH ARE GOVERNED BY AN ANALOGOUS EQUATION

The governing equations

12. In this section it will be shown that some practically important types of electromagnetic oscillation (our example has some interest in relation to the design of electron tubes) are governed by an equation having the form of (1), therefore can be investigated (with an accuracy sufficient for normal requirements) by the technique of the preceding section.

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In free space, the components H_x, H_y, H_z of the magnetic field strength H and the components E_x, E_y, E_z of the electric field strength E are related by Maxwell's equations

$$\left.\begin{aligned} \epsilon\frac{\partial}{\partial t}(E_x, E_y, E_z) &= \left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right),\quad \left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right),\quad \left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right), \\ \text{and}\qquad -\mu\frac{\partial}{\partial t}(H_x, H_y, H_z) &= \left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right),\quad \left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right),\quad \left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right), \end{aligned}\right\} \tag{12}$$

in which ϵ and μ denote respectively the dielectric constant and the permeability.* When everything is invariant with respect to z, the first three of these equations simplify to

$$\epsilon\frac{\partial}{\partial t}(E_x, E_y, E_z) = \frac{\partial H_z}{\partial y},\quad -\frac{\partial H_z}{\partial x},\quad \left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right), \tag{13}$$

and the last three simplify to

$$-\mu\frac{\partial}{\partial t}(H_x, H_y, H_z) = \frac{\partial E_z}{\partial y},\quad -\frac{\partial E_z}{\partial x},\quad \left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right). \tag{14}$$

We may distinguish (and superpose in any proportions) two particular solutions of (13) and (14): an 'electric' type of oscillation, or 'E-wave', in which $H_z = 0$; and a 'magnetic' type of oscillation, or 'H-wave', in which $E_z = 0$. In the (two-dimensional) 'E-wave'

$$\left.\begin{aligned} E_x = E_y = 0,\quad \epsilon\frac{\partial}{\partial t}E_z &= \frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}, \\ \text{and}\qquad -\mu\frac{\partial}{\partial t}(H_x, H_y) &= \frac{\partial E_z}{\partial y},\quad -\frac{\partial E_z}{\partial x}; \end{aligned}\right\} \tag{15}$$

in the 'H-wave'

$$\left.\begin{aligned} H_x = H_y = 0,\quad -\mu\frac{\partial}{\partial t}H_z &= \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}, \\ \text{and}\qquad \epsilon\frac{\partial}{\partial t}(E_x, E_y) &= \frac{\partial H_z}{\partial y},\quad -\frac{\partial H_z}{\partial x}. \end{aligned}\right\} \tag{16}$$

Eliminating H_x, H_y from (15), we have

$$\epsilon\mu\frac{\partial^2}{\partial t^2}E_z = \nabla^2 E_z, \tag{17}$$

and eliminating E_x, E_y from (16), we have

$$\epsilon\mu\frac{\partial^2}{\partial t^2}H_z = \nabla^2 H_z, \tag{18}$$

so that on the assumption that E_z or $H_z \propto \sin(pt+\epsilon)$, we have

$$[\epsilon\mu p^2 + \nabla^2]\,E_z = 0, \tag{19}$$

replacing (17), and an exactly similar equation in H_z.

* On this notation cf. Llewellyn 1941, Chap. 1.

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Again, l denoting the 'wave-length' of the oscillation, we have

$$p = 2\pi c/l, \tag{20}$$

where c denotes the velocity of light; also

$$\epsilon\mu = 1/c^2. \tag{21}$$

Hence we obtain, finally, equations of the form of (1), viz.

$$[\nabla^2+\lambda]\,(E_z, H_z) = 0,$$

where, in this electromagnetic application,

$$\lambda = \frac{4\pi^2}{l^2}. \tag{22}$$

13. The boundary conditions, on the other hand, differ for the two kinds of wave as occurring in a cylindrical conductor with axis directed along Oz. On the conducting surface the electric field-strength can have no gradient,* therefore E_z must vanish and the boundary condition for the 'E-wave' is of the kind which we considered in our acoustical Section I. But in the 'H-wave' the same requirement demands that

$$E_y \cos(x, \nu) - E_x \sin(x, \nu) = 0,$$

ν denoting the normal to the conducting surface; and hence, according to (16), we have

$$\left[\cos(x,\nu)\frac{\partial}{\partial x} + \sin(x,\nu)\frac{\partial}{\partial y}\right] H_z = \frac{\partial H_z}{\partial \nu} = 0, \tag{23}$$

i.e. *the normal gradient of H_z must vanish at the boundary.*

Examples 3 *and* 4. *Resonator system of a Klystron tube*

14. We now consider natural modes and frequencies for a conducting tube having the cross-section indicated in figure 3—an example typifying the resonator system of a 'Klystron tube'. In the actual tube, an electron beam passes in the direction of the arrow, and electromagnetic oscillations are excited in the resonator (or may be taken out of it) with the help of the coupling loop ($C.\ L.$). There are several design problems which solutions such as follow can help to solve,—e.g. gap position, gap width (d) or distance (c) for maximum efficiency. For design it is important to know the position of the current nodes in the internal surface, also how big the coupling loop must be made, and where placed, to embrace a maximum amount of flux with minimum surface loss.

Obviously energy can be given up or taken out by an electron beam only if an electric field is built up in the direction of the axis marked by an arrow, and this consideration focuses practical interest on the 'H-wave'. Since an electric field E_x is required, the x-axis must be a nodal line for H_z. Our main concern is to know the lowest frequency

* The conductivity, as is usual, is treated as infinite.

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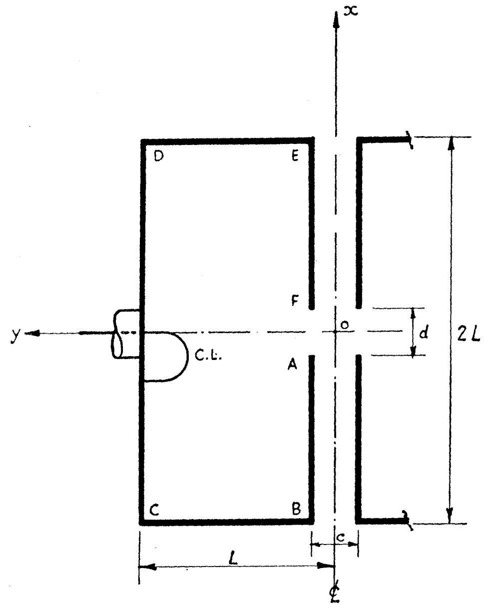

FIGURE 3

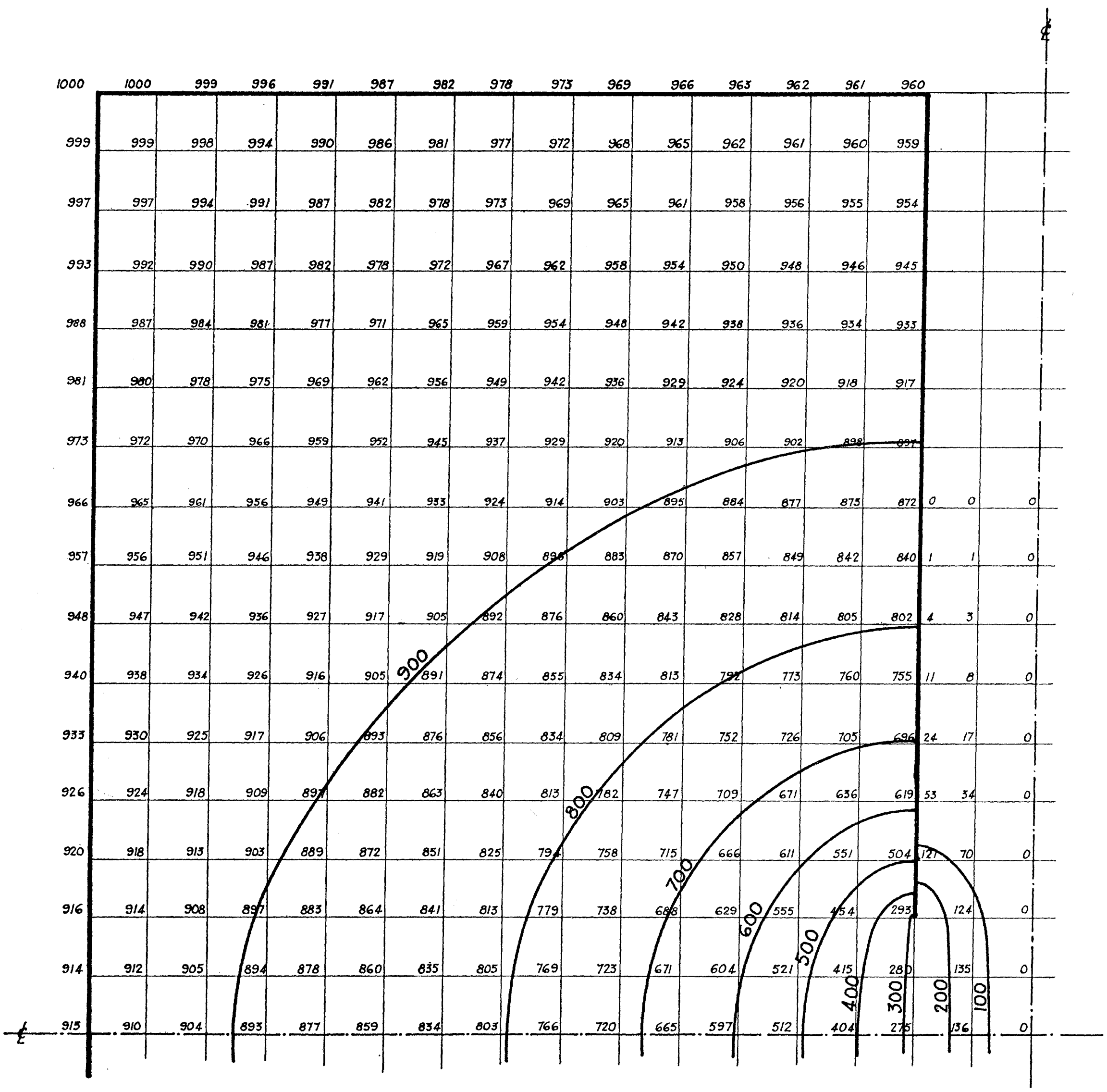

FIGURE 4

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of oscillation under these conditions. We neglect the space charge and the magnetic screening effect of the electron beam itself.

15. Interest thus attaches *by analogy* to the modes of free vibration, having a nodal line along Ox, of a membrane bounded by the surface $ABCDEF$ in figure 3, and subject at that surface to the boundary condition

$$\frac{\partial \mathbf{w}}{\partial \nu} = 0. \tag{24}$$

The system of course is capable of vibration in an indefinite number of different modes, and practical considerations will indicate which are of special interest. Usually interest centres in the gravest mode of the type described. Figure 4 presents the relaxation solution of this problem (example 3), performed by D. N. de G. A.

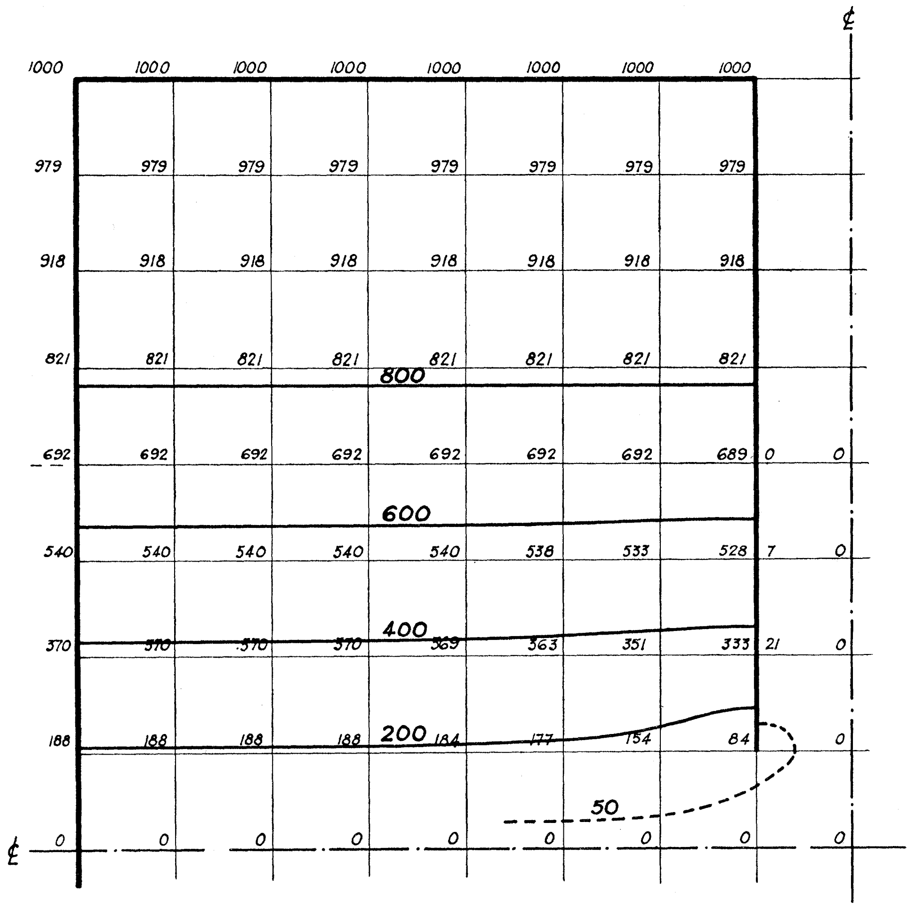

FIGURE 5

16. On account of the symmetry about Oy in figure 3 it was evident that the modes would fall into two classes, the first symmetrical and the second antisymmetical about this line. Example 3 relates to the gravest mode of the first class: a similar computation for the gravest mode of the second class (example 4) was undertaken by H. M. with results which are presented in figure 5.

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There is little in the two solutions that calls for notice from a computational standpoint, and discussion of electrical aspects is reserved. All contours meet the conducting surface orthogonally, in virtue of the condition (24). The values 0·6190 in example 3, 2·582 in example 4, were obtained for λL^2, λ being defined as in (22), and L having the significance shown in figure 3.

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